Journal of Applied Mathematics and Mechanics 74 (2010) 76–83

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

# The stability of some classes of systems in standard Bogolyubov form☆

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ARTICLE INFO

*Article history:*
Received 21 April 2009

ABSTRACT

The stability of the equilibrium of quasilinear systems in standard Bogolyubov form is investigated. Classes of systems are distinguished out for which it is possible to determine the threshold value $\varepsilon_0$ of the small parameter $\varepsilon$ which ensures qualitative agreement between the solutions of the initial systems of equations and the solutions of the averaged system corresponding it in an infinite time interval when $\varepsilon < \varepsilon_0$.



## 1. Introduction

It is well known[1,2] that Bogolyubov's second theorem provides the conditions under which both the qualitative as well as the quantitative correspondence between the solutions of the initial system of equations

$$\frac{d\mathbf{x}}{dt} = \varepsilon\mathbf{X}(t,\mathbf{x}) \tag{1.1}$$

and the solutions of the averaged system corresponding to it

$$\frac{d\mathbf{y}}{dt} = \varepsilon\mathbf{Y}(\mathbf{y}) \tag{1.2}$$

holds, in an infinite time interval, for sufficiently small

$$\mathbf{Y}(\mathbf{y}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{X}(t,\mathbf{y})dt \tag{1.3}$$

holds, in an infinite time interval, for sufficiently small values of the numerical parameter $\varepsilon$.For applications, it is of interest to have an estimate for the small parameter $\varepsilon$, that is, in each specific case it is very important to determine the threshold value $\varepsilon_0$ of the small parameter for which satisfaction of the condition $\varepsilon < \varepsilon_0$ guarantees correspondence between the solutions of the initial and averaged systems.

It turns out that it is possible to obtain such an estimate for several classes of system of the form of (1.1) by the approaches proposed earlier[3,4] using Lyapunov's second method. Unfortunately, these classes are narrower compared with those considered in Bogolyubov's second theorem.

## 2. The stability of the equilibrium of a system of the form (1.1) when the vector function X(t,x) is periodic in *t*

We will consider the case when system (1.1) can be represented in the form

$$\frac{d\mathbf{x}}{dt} = \varepsilon\left[A(t)\mathbf{x} + \mathbf{F}(t,\mathbf{x})\right] \tag{2.1}$$

☆ *Prikl. Mat. Mekh.* Vol. 74, No. 1, pp. 108-118, 2010.
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doi:10.1016/j.jappmathmech.2010.03.008

where

$$A(t) \in C_t^0, \quad \mathbf{F}(t,\mathbf{x}) \in C_{t\mathbf{x}}^{0,1}(R \times D_{\mathbf{x}}), \quad \|\mathbf{F}(t,\mathbf{x})\| = o(\|\mathbf{x}\|)$$

Furthermore, we shall assume that the matrix $A(t)$ depends $2\pi$-periodically on $t$. As regards the vector function $\mathbf{F}(t,\mathbf{x})$, only its boundedness with respect to $t\in]-\infty,\infty[$ is sufficient, that is, $2\pi$-periodicity of $\mathbf{F}(t,\mathbf{x})$ with respect to $t$ is not required.

As we see, with the assumption's which have been made regarding the vector function $\mathbf{X}(t,\mathbf{x})$, the point $\mathbf{x}=\mathbf{0}$ is an equilibrium of system (2.1).

Without loss of generality, we shall assume below that all the coefficients of system (2.1) as well as the time $t$ are dimensionless quantities.

We further use the notation

$$A^0 = \langle A \rangle = \frac{1}{2\pi}\int_0^{2\pi} A(t)dt \tag{2.2}$$

Moreover, we will represent the matrix $A(t)$ in the form

$$A(t) = A^0 + A_1(t); \quad A_1(t) = A(t) - A^0, \quad \langle A_1(t) \rangle = 0 \tag{2.3}$$

Finally, suppose $A_1^*(t)$ is the primitive of the matrix $A_1(t)$ such that $\langle A_1^*(t)\rangle = 0$.

Taking account of the above assumptions and the notation introduced, we will rewrite Eq. (2.1) in the form

$$\frac{d\mathbf{x}}{dt} = \varepsilon\left[(A^0 + A_1(t))\mathbf{x} + \mathbf{F}(t,\mathbf{x})\right] \tag{2.4}$$

Together with Eq. (2.4), we consider the auxiliary system

$$\frac{d\mathbf{x}}{dt} = A^0\mathbf{x} \tag{2.5}$$

We will denote the roots of the characteristic equation

$$|E\lambda - A^0| = 0 \tag{2.6}$$

by $\lambda_1,\ldots,\lambda_n$, where $E$ is the identity matrix.

We next distinguish the cases when the real parts of all the roots of Eq. (2.6) are negative and when a positive real part corresponds to just one root.

We will initially assume that the real parts of all the roots of Eq. (2.6) are negative, that is,

$$\operatorname{Re}\lambda_i < 0, \quad i = 1,2,\ldots,n \tag{2.7}$$

Then, by Lyapunov's theorem[5] (also, see Ref. 6), a positive definite quadratic form

$$V_0 = \mathbf{x}^T B_1 \mathbf{x}, \quad B_1^T = B_1 \tag{2.8}$$

exists whch satisfies the equation

$$L(V_0) = -U_1(\mathbf{x}) \tag{2.9}$$

in which the Lyapunov operator $L(V_0)$ is defined by the equality

$$L(V_0) = \mathbf{x}^T[A^{0T}B_1 + B_1A^0]\mathbf{x} \tag{2.10}$$

and $U_1(\mathbf{x})=\mathbf{x}^T C_1 \mathbf{x}$ is a positive-definite quadratic form.

We will now consider the regular bundles of the quadratic forms[7]

$$\mathbf{x}^T D_1\mathbf{x} - \gamma V_0, \quad -\mathbf{x}^T D_2\mathbf{x} - \mu U_1 \tag{2.11}$$

We will denote their characteristic numbers by $\gamma$

$$D_1 = A_1^{*T}B_1 + B_1A_1^*, D_2 = A^T D_1 + D_1 A \tag{2.12}$$

We will denote their characteristic numbers by $\gamma_i(t)$ and $\mu_i(t)$ ($i$=1,2,...,$n$) and we will determine these from the corresponding characteristic equations:

$$|D_1 - \gamma B_1| = 0, \quad |-D_2 - \mu C_1| = 0 \tag{2.13}$$

Suppose

$$\gamma^+ = \sup(\gamma_1(t),\ldots, \quad \gamma_n(t)), \quad \mu^+ = \sup(\mu_1(t),\ldots,\mu_n(t)), \quad \delta = \max(\gamma^+,\mu^+) \tag{2.14}$$

We now turn to the case when Eq. (2.6) admits of just one root $\lambda^*$ with a positive real part, that is, $\operatorname{Re}\lambda^*>0$. Then, according to Lyapunov's theorem, a quadratic form

$$W_0 = \mathbf{x}^T B_2\mathbf{x}; \quad B_2^T = B_2 \tag{2.15}$$

exists which satisfies the equation

$$L(W_0) = \kappa W_0 + U_2(\mathbf{x}) \tag{2.16}$$

The Lyapunov operator $L(W_0)$ on the left-hand side of Eq. (2.16) is determined by an equality of the form (2.10) with the matrix $B_1$ in it replaced by the matrix $B_2$. On the right-hand side of equality (2.16), the quantity $\kappa$ denotes a positive constant, $U_2(\mathbf{x})=\mathbf{x}^T C_2 \mathbf{x}$ is a positive definite quadratic form and, at the same time, the structure of the quadratic form is such that the set $W_0>0$ is a non-empty set.

We now consider the regular bundle of the quadratic forms

$$\beta^{-1}\mathbf{x}^T L_1\mathbf{x} - \tilde{\gamma}\mathbf{x}^T E\mathbf{x}; \quad L_1 = A_1^{*T}B_2 + B_2A_1^* \tag{2.17}$$

where $\beta$ is the largest eigenvalue of the matrix $B_2$. Since the positiveness set of the quadratic form $W_0$ is a non-empty set, it is natural that $\beta > 0$. Together with the bundle (2.17), we also consider the regular bundle of the quadratic forms

$$\mathbf{x}^T(L_2 - \kappa L_1)\mathbf{x} - \tilde{\mu}U_2; \quad L_2 = A^T L_1 + L_1 A \tag{2.18}$$

We will denote the characteristic numbers of the bundles considered by $\tilde{\gamma}_i(t)$ and $\tilde{\mu}_i(t)(i = 1, 2, \dots, n)$ and determine these from the characteristic equations

$$|\beta^{-1}L_1 - \tilde{\gamma}E| = 0, \quad |(L_2 - \kappa L_1) - \tilde{\mu}C_2| = 0 \tag{2.19}$$

We next use the notation

$$\tilde{\gamma}^+ = \sup(\tilde{\gamma}_1(t),\dots, \ \tilde{\gamma}_n(t)), \quad \tilde{\mu}^+ = \sup(\tilde{\mu}_1(t),\dots, \ \tilde{\mu}_n(t)), \quad \tilde{\delta} = \max(\tilde{\gamma}^+,\tilde{\mu}^+) \tag{2.20}$$

and formulate the basic assertion, which essentially rests on properties of the roots of the characteristic equation (2.6) and the properties of the characteristic numbers of the corresponding bundles of the quadratic forms.

**Theorem 1.** The equilibrium position $\mathbf{x} = 0$ of system (2.1) is asymptotically unstable if condition (2.7) is satisfied and $\varepsilon < \varepsilon_0$, where $\varepsilon_0=\delta^{-1}$.

Conversely, the equilibrium position $\mathbf{x} = 0$ is unstable when the characteristic equation (2.6) has just one root $\lambda^*$ with a positive real part and $\varepsilon < \varepsilon_0$, where $\varepsilon_0 = \tilde{\delta}^{-1}$.

**Proof.** We assume that condition (2.7) is satisfied and consider the auxiliary function

$$V = V_0 - \varepsilon V_1; \quad V_1 = \mathbf{x}^T\left(A_1^{*T}B_1 + B_1A_1^*\right)\mathbf{x} \tag{2.21}$$

The function $V_0$ is determined by equality (2.8). Calculating the derivative of the function $V$ with respect to $t$ along the vector field defined by system (2.4), taking account of equalities (2.8) - (2.10), we obtain

$$\frac{dV}{dt} = \varepsilon^2\left(-\mathbf{x}^T D_2\mathbf{x} - \frac{1}{\varepsilon}U_1\right) + o(\|\mathbf{x}\|^2) \tag{2.22}$$

We now rewrite equality (2.21) in the form

$$V = -\varepsilon\left(V_1 - \frac{1}{\varepsilon}V_0\right) \tag{2.23}$$

If we replace the factor $1/\varepsilon$ in the circular brackets on the right-hand sides of equalities (2.23) and (2.22) by $\gamma$ and $\mu$ respectively, we obtain the regular bundles of the quadratic forms in the form (2.11). The function $V$ will therefore be positive definite when $1/\varepsilon>\gamma^+$ and the derivative $dV/dt$ will be negative definite when $1/\varepsilon>\mu^+$. Turning to the determination of the quantity $\delta$, we see that the inequality $\varepsilon<\varepsilon_0$, provided that $\varepsilon_0=\delta^{-1}$, does, in fact, ensure the asymptotic stability of the equilibrium position $\mathbf{x} = 0$ of system (2.1). Hence, the first part of the theorem is proved.

We will now prove the second part of the theorem, assuming that a root of the characteristic equation (2.6) exists for which $\mathrm{Re}\lambda^*>0$.

We consider the auxiliary function

$$W = W_0 - \varepsilon V_2; \quad V_2 = \mathbf{x}^T\left(A_1^{*T}B_2 + B_2A_1^*\right)\mathbf{x} \tag{2.24}$$

The function $W_0$ is defined by equality (2.15). Calculating the derivative of the function $W$ with respect to $t$ along the vector field determined by system (2.4), taking account of equalities (2.15) and (2.16), we obtain

$$\frac{dW}{dt} = \varepsilon\kappa W - \varepsilon^2\left[(\mathbf{x}^T L_2\mathbf{x} - \kappa V_2) - \frac{1}{\varepsilon}U_2\right] + o(\|\mathbf{x}\|^2)$$

We see that the regular bundle of quadratic forms of the form (2.18) is contained in the square brackets on the right-hand side of this equality. Therefore, if $1/\varepsilon > \tilde{\mu}^+$, the expression

$$\varepsilon U_2 - \varepsilon^2(\mathbf{x}^T L_2\mathbf{x} - \kappa V_2)$$

becomes a positive definite function. On the other hand, a ray exists in which the equality

$$W_0 = \beta \mathbf{x}^T E \mathbf{x}$$

holds, and, therefore, if $1/\varepsilon > \tilde{\gamma}^+$, the positiveness set of the function $W$ is non-empty.

So, by choosing $\varepsilon_0 = \tilde{\delta}^{-1}$ in the case being considered, when $\varepsilon<\varepsilon_0$ we attain satisfaction of the conditions of Chetayev's theorem on instability.

*Corollary*. Under the conditions of Theorem 1, the stability (instability) of the equilibrium $\mathbf{x} = 0$ of system (2.1) is completely determined by the stability (instability) of the equilibrium $\mathbf{y} = 0$ of the system

$$\frac{d\mathbf{y}}{dt} = \varepsilon\left[A^0\mathbf{y} + \mathbf{F}(t,\mathbf{y})\right]$$

*Example* 1. We will now apply Theorem 1 to the problem of the oscillations of a pendulum with a periodically changing gravitational field and a resistant medium (Ref. 8, p. 28). In this case, the equation of motion within the specifications has the form

$$\ddot{x} + \alpha\dot{x} + \theta^2(1 + \beta\cos\omega t)x = o(\|(x,\dot{x})\|) \tag{2.25}$$

where $\alpha,\beta,\theta,\omega$ are positive constants. As above, without any loss in generality we shall assume that these constants, as well as the time $t$, are dimensionless quantities. We will investigate the stability of the trivial solution $x = \dot{x} = 0$ corresponding to the lower equilibrium position of the pendulum.

Then, assuming that the periodically changing gravitational field is a high frequency field and introducing the new independent variable $\omega t=\tau$, we rewrite Eq. (2.25) in the form of a system of two equations in the standard form

$$x' = \varepsilon\theta y, \quad y' = -\varepsilon\left[\alpha y + \theta(1 + \beta\cos\tau)x + o(\|(x,y)\|)\right]$$

where $\varepsilon=1/\omega$ and a prime denotes differentiation with respect to the variable $t$.

In this case, the auxiliary system of the type of (2.5) has the form

$$x' = \theta y, \quad y' = -(\alpha y + \theta x) \tag{2.26}$$

Since the stability of the lower equilibrium position of the pendulum is being discussed, in the case considered the real parts of the roots of the corresponding characteristic equation are negative. In the case considered, we will take the Lyapunov type equation (2.9) in the form

$$\mathbf{x}^T[A^{0T}B_1 + B_1A^0]\mathbf{x} = -\frac{\alpha}{2}(x^2 + y^2) \tag{2.27}$$

Note that there is arbitrariness in the choice of the positive definite function $U_1$ in Eq. (2.9), and, depending on in which form we select the function $U_1$. we obtain the corresponding form of the matrix $B_1$. Depending on how the function $U_1$ is chosen when forming Eq. (2.7), we have

$$B_1 = \begin{Vmatrix} 1/2 + \hat{\alpha}^2 & \hat{\alpha}/2 \\ \hat{\alpha}/2 & 1/2 \end{Vmatrix}; \quad \hat{\alpha} = \frac{\alpha}{2\theta}$$

In the case considered, the characteristic equations of the type (2.13) correspondingly have the form

$$\begin{gathered} \gamma^2\left(1 + \hat{\alpha}^2\right) - \beta^2\theta^2\sin^2\tau = 0 \\ [\beta\theta(1 + \beta\cos\tau)\sin\tau + \hat{\alpha}\mu](\beta\theta\sin\tau - \hat{\alpha}\mu) = 0 \end{gathered} \tag{2.28}$$

From Eqs (2.28) we have

$$\gamma^+ < \beta\theta, \quad \mu^+ < \frac{1}{\hat{\alpha}}(1 + \beta)\beta\theta$$

Noting that the coefficient $\alpha$, which characterizes the resistance of the medium, is usually small compared with the quantity $\theta$, we conclude that, when

$$\frac{1}{\varepsilon} \geq \frac{2\theta^2}{\alpha}(1 + \beta)\beta$$

the lower equilibrium position of the pendulum is asymptotically stable. Here, instead of the exact values for $\gamma^+$ and $\mu^+$, which are determined from Eqs (2.28), we will confine ourselves to their rougher estimates. In the problem considered, the equilibrium position of the averaged system is always asymptotically stable. However, in order to guarantee the asymptotic stability of the initial system, the frequency of the periodically changing gravitational field has to be subjected to the constraint

$$\omega \geq \frac{2\theta^2}{\alpha}(1 + \beta)\beta \tag{2.29}$$

The latter constraint becomes natural if account is taken of the fact that, in the case of a sufficiently small coefficient $\alpha$, the system considered is subjected to parametric "swinging". The choice of the frequency $\omega$ in accordance with condition (2.29) enables one to avoid this phenomenon.

*Example* 2. We will now consider the well known problem of stabilizing the upper equilibrium position of the pendulum[9,10]. The equations of motion in the case considered have the form[1,2]

$$\ddot{\theta} + \lambda\dot{\theta} + \frac{g - a\omega^2\sin\omega t}{l}\sin\theta = 0$$

Here, the notation[1,2] adopted earlier is largely retained. It is well known [1,2] that this equation can be represented in the standard form

$$\varphi' = \varepsilon\Omega + \varepsilon^2(\ldots)$$

$$\Omega' = \varepsilon\{-\sin^2\tau\sin\varphi\cos\varphi - k^2\sin\varphi + \Omega\cos\tau\cos\varphi - 2\alpha\Omega + 2\alpha\cos\tau\sin\varphi\} + \varepsilon^2(\ldots)$$

Considering the stability of the upper position of the pendulum ($\varphi=\pi$) and only retaining terms having the first power of the small parameter $\varepsilon$ as a factor, we arrive at the equations of the perturbed motion in the form

$$x_1' = \varepsilon x_2, \quad x_2' = \varepsilon\left\{\left(-\chi + \frac{1}{2}\cos 2\tau - 2\alpha\cos\tau\right)x_1 - (2\alpha + \cos\tau)x_2 + O(\mathbf{x}^2)\right\}$$

$$\chi = \frac{1}{2} - k^2, \quad \mathbf{x} = (x_1, x_2)^T \tag{2.30}$$

In this case, the auxiliary system of the type of (2.5) has the form

$$x_1' = x_2, \quad x_2' = -\chi x_1 - 2\alpha x_2$$

The real parts of the roots of the corresponding characteristic equation will be negative if

$$k^2 < 1/2 \tag{2.31}$$

In the case considered, we choose the Lyapunov type equation (2.9) in the form

$$\mathbf{x}^T[A^{0T}B_1 + B_1A^0]\mathbf{x} = -\alpha\left[\chi x_1^2 + x_2^2\right] \tag{2.32}$$

According to the choice of the function $U_1$, made when setting up Eq. (2.32), we have

$$B_1 = \begin{Vmatrix} \alpha^2 + \chi/2 & \alpha/2 \\ \alpha/2 & 1/2 \end{Vmatrix}$$

In the case considered, the characteristic equations of the type of (2.13) correspondingly have the form

$$\gamma^2 + (2\sin\tau)\gamma - \left(\alpha\sin\tau - \frac{1}{4}\sin 2\tau\right)^2(\alpha^2 + \chi)^{-1} = 0$$

$$\alpha^2\chi\mu^2 + \alpha(a_{11} + \chi a_{22})\mu + a_{11}a_{22} - a_{12}^2 = 0 \tag{2.33}$$

where

$$a_{11} = \left(\chi - \frac{1}{2} + \sin^2\tau + 2\alpha\cos\tau\right)\left(3\alpha\sin\tau - \frac{1}{4}\sin 2\tau\right)$$

$$a_{12} = \frac{1}{8}(14\alpha - \cos\tau)\sin 2\tau + \left(\alpha^2 + \chi - \frac{1}{2} + \sin^2\tau\right)\sin\tau$$

$$a_{22} = \alpha\sin\tau + \frac{5}{4}\sin 2\tau$$

From Eqs. (2.33), we determine

$$\gamma^+ = \max\left\{-\sin\tau \pm \sqrt{\sin^2\tau + \left(\alpha\sin\tau - \frac{1}{4}\sin 2\tau\right)^2(\alpha^2 + \chi)^{-1}}\right\}$$

$$\mu^+ = \frac{1}{2\alpha\chi}\max\left\{-\chi_+ \pm \sqrt{\chi_-^2 + 4\chi a_{12}^2}\right\} \tag{2.34}$$

where

$$\chi_\pm = a_{11} \pm \chi a_{22}$$

Unlike in Example 1, in the present example there is no possibility of obtaining simple analytical estimates for the quantities $\gamma^+$ and $\mu^+$. However, it can be concluded on the basis of inequalities (2.34) that the number $\varepsilon_0$ determined depends considerably on the damping coefficient $\alpha$ and on the margin with which the stability condition (2.31) of the averaged system is satisfied. In particular, if we take $\alpha$=1/40

$k$=1/2, then, carrying out the calculations using the MAPLE suite of programs, we obtain $\varepsilon_0 \approx 1/68$ for the problem considered. Hence, in order to guarantee the asymptotic stability of the upper equilibrium position of the pendulum, it is not sufficient to satisfy condition (2.31) and it is still necessary to ensure an appropriate choice of the threshold value of the small parameter $\varepsilon$. It is clear that the condition for asymptotic stability obtained is only sufficient and it remains an open question as to what extent the estimate for the small parameter $\varepsilon$ is overestimated.

## 3. Extension to the case of a conditionally-periodic vector function X($t$,x) with respect to $t$

Since the problem of small divisors arises when integrating conditionally-periodic functions, we shall henceforth restrict ourselves to a very narrow class of given functions.

*Definition*. We will say that a conditionally-periodic function (matrix function) $F(t)$ satisfies an additivity condition if the following equality holds (summation is henceforth carried out from $i = 1$ to $i = k$)

$$F(t) = \sum F_i(\omega_i t) \tag{3.1}$$

where each of the function (matrix functions) $F_i(\omega_i t)$ is periodic with period $2\pi/\omega_i$ where $\omega_1 > \omega_2 > \ldots > \omega_k \geq \omega^0 > 0$.

In particular, integrating a linear system with constant coefficients, the characteristic equation of which only has different pure imaginary roots, we arrive precisely at a matrix function with this structure.

Leaving the previous prerequisites regarding system (2.1) in force, with the exception of the prerequisite referring to the structure of the matrix $A(t)$, we next require that the matrix $A(t)$ should be conditionally-periodic and satisfy the additivity condition. Then, by the definition of the matrix $A(t)$ presented above, we have a representation in the form

$$A(t) = \sum A_i(\omega_i t) \tag{3.2}$$

We use the notation

$$A^0 = \langle A \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)dt$$

Naturally, in this case,

$$A^0 = \sum A_i^0; \quad A_i^0 = \langle A_i \rangle = \frac{1}{T_i} \int_0^{T_i} A_i(\omega_i t)dt, \quad T_i = \frac{2\pi}{\omega_i}$$

It is subsequently convenient to represent each of the matrices $A_i(\omega_i t)$ in the form

$$A_i(\omega_i t) = A_i^0 + \tilde{A}_i(\omega_i t); \quad \tilde{A}_i(\omega_i t) = A_i(\omega_i t) - A_i^0, \quad \langle \tilde{A}_i(\omega_i t) \rangle = 0$$

Finally, denoting the primitives of the matrices $\tilde{A}_i(\omega_i t)$ by $A_i^*(\omega_i t)$ such that $\langle A_i^*(\omega_i t) \rangle = 0$, in the case considered we reduce Eqs (2.1) to the form

$$\frac{d\mathbf{x}}{dt} = \varepsilon\left[\left(A^0 + \sum \tilde{A}_i(\omega_i t)\right)\mathbf{x} + \mathbf{F}(t,\mathbf{x})\right] \tag{3.3}$$

As previously, together with Eq. (3.3), we shall consider an auxiliary system of the form (2.5) and the characteristic equation of the form (2.6) corresponding to it.

Assuming initially that the real parts of all the roots of the characteristic equation (2.6) are negative, we consider the regular bundles of the quadratic forms

$$\mathbf{x}^T \tilde{D}_1 \mathbf{x} - \gamma \tilde{V}_0, \quad -\mathbf{x}^T \tilde{D}_2 \mathbf{x} - \mu \tilde{U}_1 \tag{3.4}$$

where

$$\tilde{D}_1 = \left(\sum A_i^*\right)^T \tilde{B}_1 + \tilde{B}_1\left(\sum A_i^*\right), \quad \tilde{D}_2 = A^T \tilde{D}_1 + \tilde{D}_1 A$$

Here, in the same as above, $\tilde{V}_0 = \mathbf{x}^T \tilde{B}_1 \mathbf{x}$ and $\tilde{U}_1(\mathbf{x}) = \mathbf{x}^T \tilde{C}_1 \mathbf{x}$ are positive definite quadratic forms for which Lyapunov's equation

$$\mathbf{x}^T[A^{0T}\tilde{B}_1 + \tilde{B}_1 A^0]\mathbf{x} = -\mathbf{x}^T \tilde{C}_1 \mathbf{x}$$

is satisfied.

The characteristic number of the bundles (3.4) are denoted by $\gamma_i(t)$ and $\mu_i(t)$ ($i$=1,2,. . .,$n$) respectively. In determining them from the characteristic equations

$$\left|\tilde{D}_1 - \gamma \tilde{B}_1\right| = 0, \quad |-\tilde{D}_2 - \mu \tilde{C}_1| = 0$$

we shall assume that

$$\gamma^+ = \sup(\gamma_1(t),\ldots,\gamma_n(t)), \quad \mu^+ = \sup(\mu_1(t),\ldots,\mu_n(t))$$

It can be seen that the bundles of the quadratic forms (3.4) have the same structure as in Section 2.

In the same way as described above, when a characteristic equation of the form of (2.6) admits of just one root $\lambda^*$ with a positive real part, we consider the regular bundles of the quadratic forms

$$\tilde{\beta}^{-1}\mathbf{x}^T\tilde{L}_1\mathbf{x} - \tilde{\gamma}\mathbf{x}^T E\mathbf{x}, \quad \mathbf{x}^T(\tilde{L}_2 - \tilde{\kappa}\tilde{L}_1)\mathbf{x} - \tilde{\mu}\tilde{U}_2 \tag{3.5}$$

where

$$\tilde{L}_1 = \left(\sum A_i^*\right)^T\tilde{B}_2 + \tilde{B}_2\left(\sum A_i^*\right), \quad \tilde{L}_2 = A^T\tilde{L}_1 + \tilde{L}_1 A$$

Here, by analogy with Section 2, $\tilde{W}_0 = \mathbf{x}^T\tilde{B}_2\mathbf{x}$ and $\tilde{U}_2 = \mathbf{x}^T\tilde{C}_2\mathbf{x}$ are quadratic forms associated with Lyapunov's equation

$$\mathbf{x}^T[A^{0T}\tilde{B}_2 + \tilde{B}_2A^0]\mathbf{x} = \tilde{\kappa}\tilde{W}_0 + \tilde{U}_2$$

which corresponds to an unperturbed system of the form of (2.5). Here, the quantity $\tilde{\kappa}$ denotes a positive constant, the quadratic form $\tilde{U}_2$ is positive definite, the positiveness set of the quadratic form $\tilde{W}_0$ is non-empty and $\tilde{\beta} > 0$ corresponds to the largest eigenvalue of the matrix $\tilde{\beta}_2$.

The characteristic numbers of the bundles considered are denoted by $\tilde{\gamma}_i(t)$ and $\tilde{\mu}_i(t)(i = 1, 2, \ldots, n)$ respectively. These numbers are determined from characteristic equations, which differ from Eqs (2.19) in that $L_1$,$L_2$,$\kappa$,$C_2$ is replaced by $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{\kappa}$, $\tilde{C}_2$. As above, we shall use the first two definitions (2.20).

**Theorem 2.** Suppose the conditionally-periodic matrix $A(t)$ in Eqs (2.1) satisfies the additivity condition. The equilibrium position $\mathbf{x} = 0$ of system (2.1) is then asymptotically stable if condition (2.7) is satisfied and $\varepsilon<\varepsilon_0$, where $1/\varepsilon_0=\max(\gamma^+,\mu^+)$.

Conversely, the equilibrium position $\mathbf{x} = 0$ is unstable when the characteristic equation (2.6) has roots with a positive real part and $\varepsilon<\varepsilon_0$, where

$$1/\varepsilon_0 = \max(\tilde{\gamma}^+,\tilde{\mu}^+) \tag{3.6}$$

*Proof.* Consider the auxiliary functions

$$V = \tilde{V}_0 - \varepsilon V_1, \quad W = \tilde{W}_0 - \varepsilon V_2 \tag{3.7}$$

the expressions for which contain the Lyapunov functions $\tilde{V}_0$ and $\tilde{W}_0$ for an auxiliary system of the form (2.5), and functions (3.7) correspond to the case of its asymptotic stability and instability respectively. As regards the terms $\varepsilon V_1$ and $\varepsilon V_2$, where

$$V_1 = \mathbf{x}^T\tilde{D}_1\mathbf{x}, \quad V_2 = \mathbf{x}^T\tilde{L}_1\mathbf{x}$$

they are correction terms in their own way if account is taken of the fact that the functions $V$ and $W$ are subsequently used as Lyapunov functions when investigating system (3.3) which will be a system with variable coefficients. It is natural that the terms $\varepsilon V_1$ and $\varepsilon V_2$ must somehow reflect the structure of system (3.3).

We calculate the derivative of the functions $V$ and $W$ along the vector field defined by system (3.3). We arrive at the equalities

$$\begin{aligned}\frac{dV}{dt} &= \varepsilon^2\left(-\mathbf{x}^T\tilde{D}_2\mathbf{x} - \frac{1}{\varepsilon}\tilde{U}_1\right) + o(\|\mathbf{x}\|^2)\\ \frac{dW}{dt} &= \varepsilon\tilde{\kappa}W - \varepsilon^2\left[(\mathbf{x}^T\tilde{L}_2\mathbf{x} - \tilde{\kappa}V_2) - \frac{1}{\varepsilon}\tilde{U}_2\right] + o(\|\mathbf{x}\|^2)\end{aligned} \tag{3.8}$$

It can be seen that the terms $\varepsilon V_1$ and $\varepsilon V_2$ in equalities (3.7) are chosen in such a way that variable terms containing the small parameter $\varepsilon$ to the first power do not appear on the right-hand sides of equalities (3.8).

Next, representing the first equality of (3.7) in the form

$$V = -\varepsilon\left(V_1 - \frac{1}{\varepsilon}\tilde{V}_0\right) \tag{3.9}$$

we see that, replacing the factor $1/\varepsilon$ in equality (3.9) and the first equality of (3.8) by $\gamma$ and $\mu$ respectively, we arrive at the pair of regular bundles of the quadratic forms (3.4). Now, applying the mode of reasoning described in Section 2, we conclude that the first part of Theorem 2 is true.

Unlike the first equality of (3.7), the right-hand side of the second equality of (3.7) does not reduce to a regular bundle of quadratic forms since, in the general case, the quadratic form $\tilde{W}_0$ is of alternating sign. Hence, when discussing instability, we consider the regular bundles of the quadratic forms in the form of the first expression of (3.5). The choice of the small parameter $\varepsilon$ in accordance with the inequality $1/\varepsilon > \tilde{\gamma}^+$ ensures a non-empty set $W > 0$. On the other hand, putting $1/\varepsilon > \tilde{\mu}^+$, we achieve the property of a negative sign for the expression in the square brackets on the right-hand side of the second equality of (3.8). It follows that, if equality (3.6) holds, the conditions of Chetayev's instability theorem are satisfied.

Theorem 2 is proved.

Under the conditions of Theorem 2, there is a corollary similar to that obtained in Section 2.

*Example* 3. By analogy with Example 1, we apply Theorem 2 to the problem of the oscillations of a pendulum in a conditionally-periodically changing gravitational field and a resistant medium. In this case, we represent the equation of motion in the form

$$\ddot{x} + \alpha\dot{x} + \theta^2(1 + \beta\sigma(t))x = o(\|(x,\dot{x})\|) \tag{3.10}$$

where, as in Example 1, $\alpha,\beta,\theta$ are positive constants. As above, without loss in generality, we shall assume that these constants, as well as the time $t$, are dimensionless quantities. Furthermore, we shall assume that the conditionally-periodic function $\sigma(t)$ satisfies the additivity condition and has the form (3.1). Without loss in generality, we will take the time average of $\sigma(t)$ to be equal to zero and investigate the trivial solution $x = \dot{x} = 0$ of the corresponding lower equilibrium position of the pendulum.

Next, assuming that the conditionally-periodically varying gravitational field is a high-frequency field and introducing the new independent variable $\omega_k t = \tau$, we rewrite Eq. (3.10) in the form of a system of two equations in standard Bogolyubov form

$$x' = \varepsilon\theta y, \quad y' = -\varepsilon\left[\alpha y + \theta(1 + \beta\sigma(\tau))x + o(\|(x, y)\|)\right]; \quad \varepsilon = 1/\omega_k \tag{3.11}$$

In this case, the auxiliary system of the type (2.5) has the form (2.26).

In the case considered, the real parts of the roots of the corresponding characteristic equation are negative. We choose a Lyapunov equation of the type (2.9) in the form

$$\mathbf{x}^T[A^{0T}B_1 + B_1A^0]\mathbf{x} = -2\theta(x^2 + y^2)$$

Hence, we have

$$B_1 = \begin{Vmatrix} 2\theta/\alpha + \alpha/\theta & 1 \\ 1 & 2\theta/\alpha \end{Vmatrix}$$

In the case considered, the characteristic equations of type (2.13) take the form

$$\begin{gathered}\gamma^2\left(4\theta^2 + \alpha^2\right) - 4\theta^4\beta^2\sigma^*(\tau)^2 = 0 \\ \left\{2\beta\theta^2[1 + \beta\sigma(\tau)]\sigma^*(\tau) + \alpha\mu\right\}\left[2\beta\theta^2\sigma^*(\tau) - \alpha\mu\right] = 0\end{gathered} \tag{3.12}$$

where $\sigma^*(\tau)$ is the primitive of the function $\sigma(t)$ such that $\langle\sigma^*(\tau)\rangle=0$.

The coefficient $\alpha$, characterizing the resistance of the medium, is usually small compared with the quantity $\theta$. From Eqs (3.12), we therefore have

$$\frac{1}{\varepsilon_0} = \frac{2\beta\theta^2}{\alpha}\delta, \quad \delta = \max(\sup|\sigma^*(\tau)[1 + \beta\sigma(\tau)]|, \quad \sup|\sigma^*(\tau)|)$$

So, when

$$\omega_k > \frac{2\beta\theta^2}{\alpha}\delta$$

the lower equilibrium position of the pendulum is asymptotically stable.

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